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System approximation using orthonormal functions

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SYSTEM APPROXIMATION USING ORTHONORMAL FUNCTIONS

by

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SYMBOLS

a_n	-	nth coefficient in approximating series
$a]$	-	column of coefficients a_1, - - - a_N
c_{nm}	-	constant defined by $\int_0^\infty \varphi_n \varphi_m \, dt$
C	-	N by N matrix with elements c_{nm}
d_n	-	constant defined by $\int_0^\infty \psi_{io} \varphi_n \, dt$
$d]$	-	column of coefficients d_1, - - - - d_N
E	-	mean square error
$f_i(t)$, $F_i(s)$	-	prescribed input
$f_o(t)$, $F_o(s)$	-	output of system under test
$h(t)$, $H(s)$	-	impulse response, transfer function of system under test
$h^*(t)$, $H^*(s)$	-	approximation of impulse response, Approximation of transfer function of system under test
k_n	-	computer circuit bias measurements
N	-	number of approximating functions
s	-	complex variable of Laplace transform
s_n	-	nth pole of $H^*(s)$
T	-	averaging time
W	-	power per unit bandwidth
α_n	-	negative real part of nth pole
β_n	-	distance of nth pole from origin
$\theta_n(t)$, $\Theta_n(s)$	-	nth approximating function in $f_o^*(t)$, $F_o^*(s)$
Λ_n	-	all pass function with poles of the nth orthonormal function
σ	-	standard deviation
τ	-	variable of integration, argument of correlation function
$\varphi_n(t)$, $\Phi_n(s)$	-	nth approximating function in $h^*(t)$, $H^*(s)$
$\psi_{ii}(t)$	-	auto-correlation function of $f_i(t)$
$\psi_{io}(t)$	-	cross-correlation function of $f_i(t)$ and $f_o(t)$

SYSTEM APPROXIMATION USING ORTHONORMAL FUNCTIONS

INTRODUCTION

Experimental determination of the dynamic characteristics of a linear system is often desirable both in the analysis and the design of control systems. The classical method of measuring a linear system transfer function is to observe system response to an artificial disturbance, such as a sinusoidal or step input. A more general and often more practical approach is made possible through the use of statistical methods, which make use of random inputs. Through the use of such inputs actual operating records may be used in determining transfer functions.

The purpose of this study is to investigate one statistical method of measuring linear system transfer functions. Based on minimization of the mean square error, a finite series will be chosen to approximate the system transfer function. The terms of this series will be a set of orthonormal functions, the coefficients of which are determined by statistical methods. However, no direct computation of correlation functions is necessary.

No attempt will be made in this work to determine the characteristics of any particular unknown system, since this is only intended to be a feasibility study of the proposed method. Two known second order systems will be investigated, by approximating their transfer functions with a set of four orthonormal functions.

In order to simplify the mathematics and mechanization, a white noise source is used as input, although the method would work equally well for any other statistically describable input with suitable generalization.

THEORY

The choice of an error measure in forming the series representation is basic to the method used. Throughout this study minimization of the mean square error will be the criterion. Other error measures, such as magnitude of error, are possible and may in some instances be preferable, but the mean square error approach allows a convenient mathematical description of the problem.

Before proceeding with the theoretical development, certain statistical measures must be defined. In a stationary random signal the time dependent signal itself cannot be defined, but a statistical measure known as its auto-correlation function can. The auto-correlation function is, mathematically,

$$\psi_{11}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f_1(t)\, f_1(t+\tau)\, dt = \text{avg } f_1(t)\; f_1(t+\tau) \tag{1}$$

Two distinct signals may be correlated using the same technique. This defines the cross-correlation function between the two signals to be

$$\psi_{12}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f_1(t)\, f_2(t+\tau)\, dt = \text{avg } f_1(t)\; f_2(t+\tau) \tag{2}$$

The transfer function of a system, or equivalently its weighting function, may be found, having only the above statistical descriptions of its input and output. This is shown in the following development, as outlined in Ref. 1. An alternate method is shown in Ref. 2.

Let $f_i(t)$ be the input and $f_o(t)$ be the output of a linear system with weighting

function h(x),

then

$$f_o(t) = \int_{-\infty}^{+\infty} h(x)\, f_i(t-x)\; dx \tag{3}$$

and

$$\psi_{io}(\tau) = \text{avg } f_i(t)\; f_o(t+\tau) = \text{avg } f_i(t-\tau)\; f_o(t) \tag{4}$$

Substituting for $f_o(t)$

$$\psi_{io}(\tau) = \text{avg } f_i(t-\tau) \int_{-\infty}^{+\infty} h(x)\, f_i(t-x)\; dx$$

Changing the order of integration

$$\psi_{io}(\tau) = \int_{-\infty}^{+\infty} h(x)\; dx \quad \text{avg } f_i(t-\tau)\; f_i(t-x)$$

But, by definition

$$\psi_{ii}(\tau - x) = \text{avg } f_i(t-\tau)\; f_i(t-x)$$

Therefore

$$\psi_{io}(\tau) = \int_{-\infty}^{+\infty} h(x)\; \psi_{ii}(\tau - x)\; dx \tag{5}$$

Solution of the convolution integral of Eq. 5 is simplified if the input signal is white noise, meaning the signal has constant power per unit bandwidth (W). In this case the auto-correlation function is an impulse of strength W at $\tau = 0$. Solution of Eq. 5 is then simply

$$\psi_{io}(\tau) = W\; h(\tau) \qquad \text{or} \qquad h(\tau) = \frac{1}{W}\, \psi_{io}(\tau) \tag{6}$$

Thus, with a white noise input the system weighting function is directly proportional to the cross correlation function, the constant of proportionality being the reciprocal of input power per unit bandwidth. The procedure is therefore reduced to determining cross-correlation between input and output signals of the system under test.

Before explaining the proposed method of accomplishing this determination, it might be well to mention some of the more widely used techniques available. Correlation functions may be calculated directly from the definition. This involves multiplication, summation and averaging over a reasonably long period of time, one of the multiplicands being delayed a certain time, τ. The MIT electronic correlator mechanizes the above procedure, calculating the correlation function at discrete values of the argument τ (Ref. 3). Either an analog or a digital computer may be used in the calculations, although a digital computer is normally preferred because of the ease in programming a delay in a multiplicand.

Some thought on the methods above shows an inherent disadvantage. Because discrete values of the argument τ must be used in calculating correlation functions, a continuous curve is not obtainable; and hence no functional description of the correlation functions is obtained by these methods.

The method considered in this study is intended to overcome these limitations. An entirely different approach is used, in which an attempt is made to obtain a series representation of the entire correlation function over all values of τ rather than obtain specific values for discrete τ. Thus the correlation function is defined functionally for any τ with arbitrarily small error depending upon the goodness of the approximating series. The measure of approximation

error using this method will not be in the number of τ values computed, but rather in how well the series approximates the true correlation function.

It was shown above that the desired system weighting function is directly proportional to the cross correlation between input and output when the input is white noise. The cross-correlation function will be approximated by

$$\psi_{io}(t) \doteq \psi_{io}^{*}(t) = \sum_{n=1}^{N} a_n \varphi_n(t) \tag{7}$$

and, from Eq. 6,

$$h(t) \doteq h^{*}(t) = \frac{1}{W} \sum_{n=1}^{N} a_n \varphi_n(t) \tag{8}$$

The frequency expression equivalent to Eq. 8 provides an approximation of the system transfer function.

$$H(s) \doteq H^{*}(s) = \frac{1}{W} \sum_{n=1}^{N} a_n \Phi_n(s) \tag{9}$$

In the above equations $\Phi_n(s)$ are approximating functions and a_n are the coefficients of these functions. The choice of approximating functions is obviously of primary importance, and should be based on what knowledge of system behavior is available. A judicious choice of $\Phi_n(s)$ will reduce the number of series terms required for a certain accuracy. No attempt will be made to set up rules or techniques for such a choice, since this study deals with method feasibility after a proper choice of $\Phi_n(s)$ has been made. A discussion of the factors involved in choosing $\Phi_n(s)$ may be found in Ref. 4. A greater number of approximating functions will of course make possible a greater accuracy, the number being limited by the complexity allowed.

Having decided upon a set of approximating functions, the coefficient of each can be found on the basis of minimum mean square error. This mean square error is

$$E = \int_0^\infty [\psi_{io}(t) - \psi_{io}^*(t)]^2 dt = \int_0^\infty [\psi_{io}(t) - \sum_{n=1}^{N} a_n \varphi_n(t)]^2 \, dt \tag{10}$$

Expanding

$$E = \int_0^\infty \psi_{io}^2(t) \, dt - 2 \sum_{n=1}^{N} a_n \int_0^\infty \psi_{io}(t) \varphi_n(t) \, dt + \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m \int_0^\infty \varphi_n(t) \varphi_m(t) \, dt$$

For simplification, define

$$d_n = \int_0^\infty \psi_{io}(t) \varphi_n(t) \, dt \quad \text{and} \quad c_{nm} = \int_0^\infty \varphi_n(t) \varphi_m(t) \, dt \tag{11}$$

Substituting

$$E = \int_0^\infty \psi_{io}^2(t) \, dt - 2 \sum_{n=1}^{N} a_n d_n + \sum_{n=1}^{N} \sum_{m=1}^{N} a_n a_m c_{nm} \tag{12}$$

To minimize the above with respect to a_n, set

$$\frac{\partial E}{\partial a_n} = 0 = 0 - 2 d_n + 2 \sum_{m=1}^{N} a_m c_{nm} \qquad n = 1, 2, \cdots N \tag{13}$$

or,

$$d_n = \sum_{m=1}^{N} c_{nm} a_m \qquad n = 1, 2, \cdots N \tag{14}$$

This is a set of N linear equations:

$$\begin{aligned} d_1 &= c_{11} a_1 + c_{12} a_2 + \cdots + c_{1N} a_N \\ &\vdots \\ d_N &= c_{N1} a_1 + c_{N2} a_2 + \cdots + c_{NN} a_N \end{aligned} \tag{15}$$

In matrix notation

$$d] = C\, a] \qquad \text{or} \qquad a] = C^{-1}\, d] \tag{16}$$

It has been shown (Ref. 4) that the minimum so obtained is unique and is the system minimum, provided both the $\Phi_n(s)$ and Eqs. 15 are linearly independent.

If the approximating functions are chosen to be orthonormal, that is

$$\int_0^{\infty} \varphi_n(t)\, \varphi_m(t)\, dt = C_{nm} = 1 \qquad n=m$$
$$= 0 \qquad n \neq m \tag{17}$$

the square matrix "C" is the identity matrix, and,

$$a_n = d_n = \int_0^{\infty} \psi_{io}(t)\, \varphi_n(t)\, dt \tag{18}$$

The mean square error is then

$$E_{min} = \int_0^{\infty} \psi_{io}^2(t)\, dt - \sum_{n=1}^{N} a_n^2 \tag{19}$$

This error is due solely to the finite series approximation and will always be present when using this method.

Eq. 18 is the theoretical basis for analog computer determination of the finite series coefficients. Substituting in this equation the expression for the cross-correlation function from Eq. 4 yields

$$a_n = \frac{1}{T} \int_0^{T} \int_0^{\infty} f_i(t-\tau)\, f_o(t)\, \varphi_n(\tau)\, d\tau\, dt \tag{20}$$

Note that T is not permitted to approach infinity in the above equation. For the remainder of this development a finite averaging time will be assumed. Interchanging the order of integration,

$$a_n = \frac{1}{T}\int_0^T f_o(t)\left[\int_0^\infty f_i(t-\tau)\,\varphi_n(\tau)\,d\tau\right]dt$$

Let $\theta_n(t)$ represent the output of a linear system with impulse response $\varphi_n(t)$ and input f_i, then,

$$a_n = \frac{1}{T}\int_0^T f_o(t)\;\theta_n(t)\;dt \tag{21}$$

in which

$$\theta_n(t) = \int_0^\infty f_i(t-\tau)\;\varphi_n(\tau)\;d\tau \tag{22}$$

Eq. 21 can be mechanized as shown in Fig. 1, a white noise source being used. The figure shows the method of obtaining one coefficient; others could be obtained in the same manner. Since all series coefficients can be determined by the computer mechanization, and since the $\Phi_n(s)$ have been chosen, no additional mathematical development is necessary in determining the finite series.

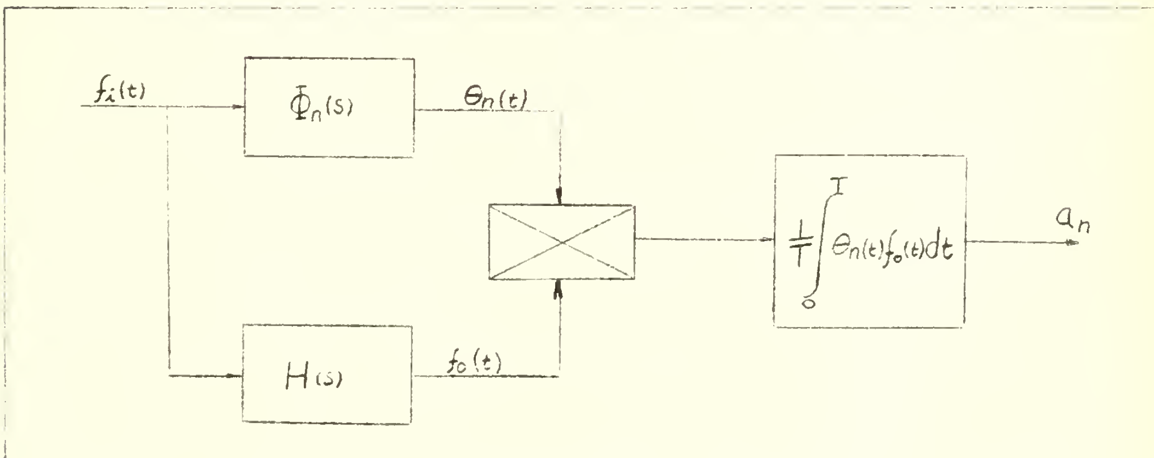

FIGURE 1
Measurement of Orthonormal Function Coefficients

Although the above analysis has been restricted to the time domain, the results are available in the frequency domain as well. By Eq. 9, using the same a_n found above to multiply $\Phi_n(s)$, an approximation to the system transfer function is immediately available.

After the coefficients have been determined, each of the orthonormal approximating functions of the computer circuit can be multiplied by its proper coefficient, and thus the system under test can be simulated. That is, an impulse can be applied to the set of approximating functions, their outputs summed and plotted to form a graphical representation of the approximate system weighting function. This procedure is illustrated in Fig. 2.

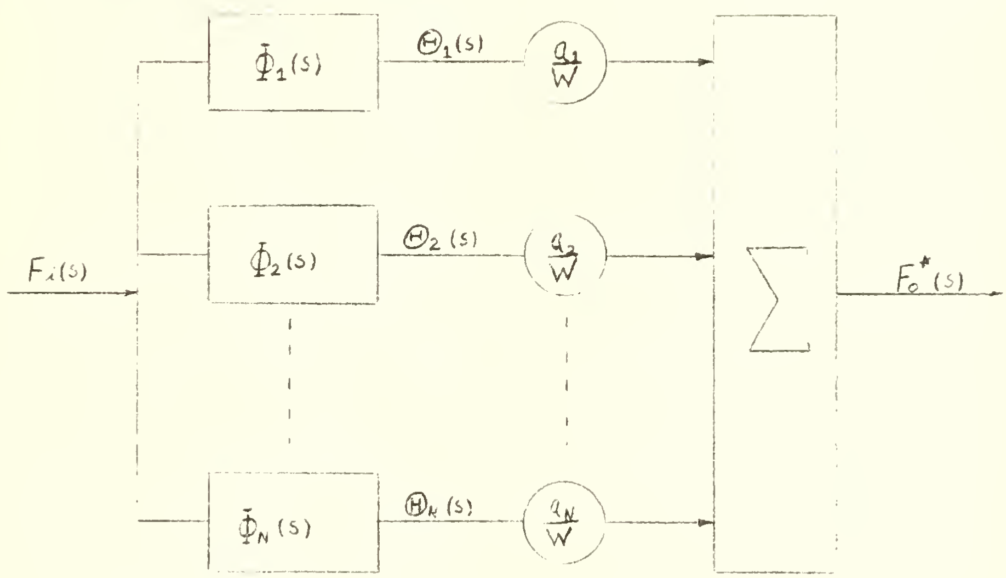

FIGURE 2

Simulation of H * (s)

ORTHONORMAL FUNCTIONS

As orthonormal functions greatly simplify coefficient determination, it would be desirable to have some means of obtaining a set of these functions that would not only permit a wide selection of approximating functions but also be amenable to simulation by analog equipment.

By using the theory of residues, Ref. 3 develops a set of orthonormal functions that meet these requirements. Starting with a set of linearly independent transfer functions, $\overline{\Phi_n(s)}$, it is possible to obtain a set of functions, $\Phi_n(s)$, such that, in addition to orthonormality,

$$\begin{aligned}
\Phi_1(s) &= f_1\left(\overline{\Phi_1(s)}\right) \\
\Phi_2(s) &= f_2\left(\overline{\Phi_1(s)}, \overline{\Phi_2(s)}\right) \\
\Phi_3(s) &= f_3\left(\overline{\Phi_1(s)}, \overline{\Phi_2(s)}, \overline{\Phi_3(s)}\right) \\
&\vdots \\
\Phi_n(s) &= f_n\left(\overline{\Phi_1(s)}, \overline{\Phi_2(s)}, \overline{\Phi_3(s)}, \cdots \overline{\Phi_n(s)}\right)
\end{aligned} \tag{23}$$

If the linearly independent functions, $\overline{\Phi_n(s)}$, have real poles at $s_n = -\alpha_n$, and complex poles at

$$s_{n,n+1} = -\alpha_n \pm j\sqrt{\beta_n^2 - \alpha_n^2}$$

the orthonormal functions become

$$\Phi_n(s) = \Lambda_{n-1}(s)\,\frac{\sqrt{2\alpha_n}}{s+\alpha_n}$$

for real roots, and

$$\Phi_n(s) = \Lambda_{n-1}(s)\,\frac{2\beta_n\sqrt{\alpha_n}}{s^2+2\alpha_n s+\beta_n^2}, \quad \Phi_{n+1}(s) = \Lambda_{n-1}(s)\,\frac{2\sqrt{\alpha_n}\,s}{s^2+2\alpha_n s+\beta_n^2} \tag{24}$$

for complex roots, where

$$\Lambda_{n-1}(S) = \frac{(S+S_1)(S+S_2)-------(S+S_{n-1})}{(S-S_1)(S-S_2)-------(S-S_{n-1})} \tag{25}$$

Ref. 3 also shows that the analog circuitry necessary to realize these functions is

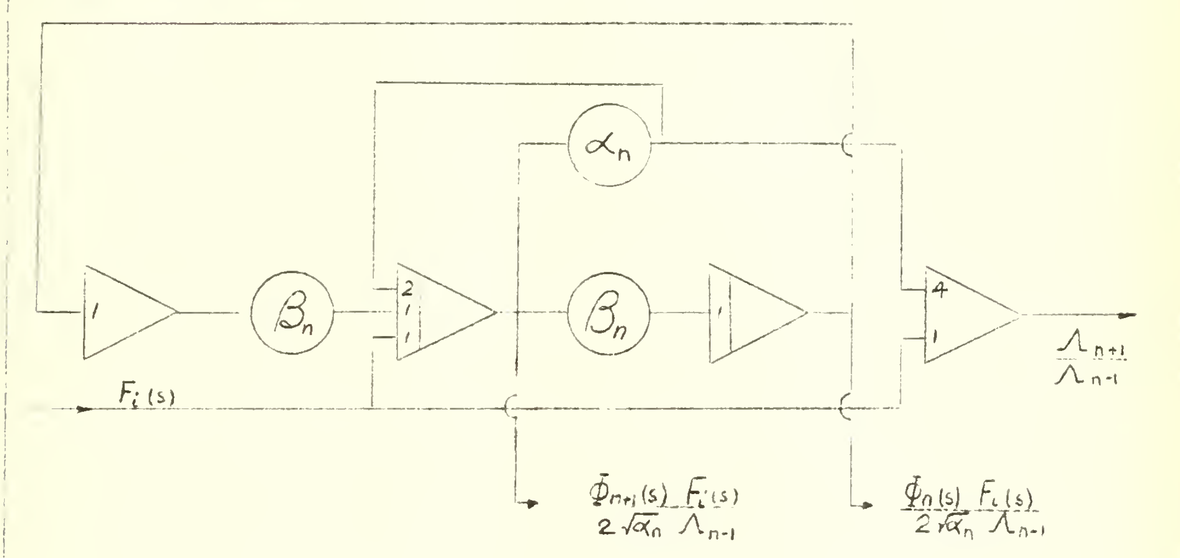

FIGURE 3
Orthonormal Function Mechanization

ERRORS

The discussion up to this point has assumed ideal components. It would be relevant at this point to consider some of the errors and equipment limitations that would be present when the system is mechanized and relatively short averaging times are used. The errors may be divided into three groups: those

due to biases in the electronic components, those due to the non-linearities of the servo multipliers, and those due to the use of finite averaging times.

Static Error Analysis

In order to study the effect of biases, the circuit shown below in Fig. 4 will be used.

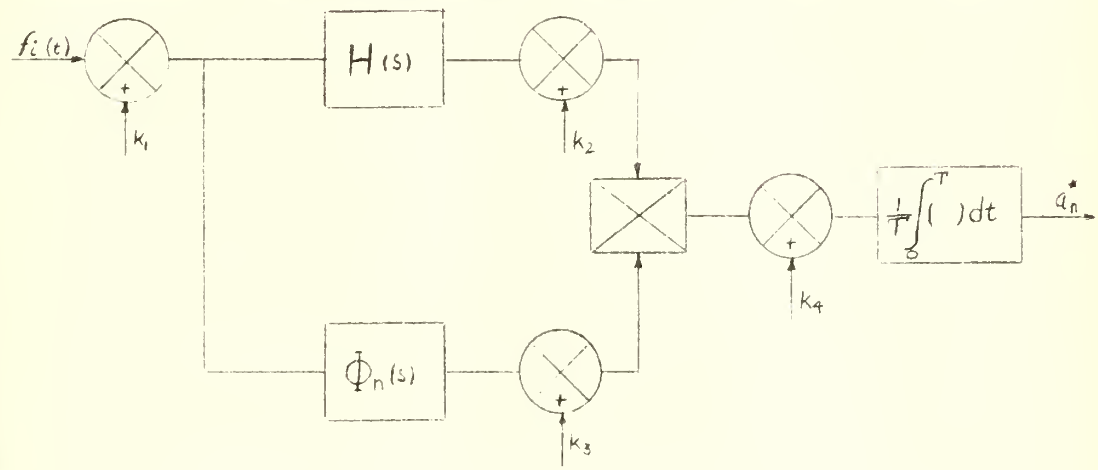

FIGURE 4
Schematic Showing Assumed Static Errors

Here k_1 may be thought of as an essentially constant bias on the input, f_i ; k_2 and k_3 as constant biases in the circuitry making up H(s) and $\Phi_n(s)$, respectively; and k_4, a constant, as the combined effect of servo multiplier static error and bias in the integrator following the multiplier. It should be noted that H(o) and $\Phi_n(o)$ are the DC responses of H(s) and $\Phi_n(s)$, respectively.

Then the output of the multiplier would be

$$(f_o + k_1 H(o) + k_2)(\theta_n + k_1 \Phi_n(o) + k_3) .$$

If a_n^* is now defined as the coefficient computed by the circuit in Fig. 4, the error in a_n^* may be evaluated by noting that

$$a_n^* = \frac{1}{T}\int_0^T \left\{ [f_o + k_1 H(o) + k_2][\theta_n + k_1 \Phi_n(o) + k_3] + k_4 \right\} dt \tag{26}$$

and

$$a_n = \frac{1}{T}\int_0^T f_o \theta_n \, dt$$

so

$$a_n^* - a_n = \frac{1}{T}\int_0^T [f_o k_1 \Phi_n(o) + f_o k_3 + \theta_n k_1 H(o) + k_1^2 H(o) \Phi_n(o) + k_2 \theta_n$$
$$k_1 k_2 \Phi_n(o) + k_1 k_3 H(o) + k_2 k_3 + k_4] dt$$

If T is large enough, the finite time average values of the random functions θ_n and f_o will be essentially zero, which gives

$$a_n^* - a_n = \frac{1}{T}\int_0^T [k_1^2 H(o) \Phi_n(o) + k_1 k_3 H(o) + k_1 k_2 \Phi_n(o) + k_2 k_3 + k_4] dt \tag{27}$$

But the average value of a constant is the constant itself, so that

$$a_n^* - a_n = k_1^2 H(o) \Phi_n(o) + k_1 k_3 H(o) + k_1 k_2 \Phi_n(o) + k_2 k_3 + k_4 \tag{28}$$

The mechanization of orthonormal functions previously described gave two types of functions; a constant divided by a quadratic in s, and a constant multiplying s divided by a quadratic in s. For the latter case, the DC response is zero, or $\Phi_n(o) = 0$, and

$$a_n^* - a_n = k_1 k_3 H(0) + k_2 k_3 + k_4 \tag{29}$$

Further application of these results will depend upon the hardware being used. k_2, k_3, and k_4 may be measured more or less directly, and $k_1 = \frac{1}{T}\int_0^T f_i \, dt$ With a properly adjusted noise source, k_1 should be of the same order of magnitude as k_2 and k_3; the use of good servo multipliers should assure that k_4 would be of the same order of magnitude as the other perturbation voltages. Thus if second order effects are neglected, Eq. 28 becomes

$$a_n^* - a_n \doteq k_4 \tag{30}$$

Servo Multiplier Limitations

The preceding analysis treated errors that were electronic in origin; the next source of error to be considered is the servo multiplier. As used in system approximation, f_o is used to drive the shaft, which positions a potentiometer arm, and θ_n is fed into the potentiometer winding.

While the multiplicand will be reproduced on the potentiometer without time delay, the multiplier, on the other hand, may be subject to both a time delay and a dynamic position error due to the inertia of the servo and the finite torque available from the motor windings.

The servo specifications are usually given in the form of a limiting position (volts), a limiting velocity (volts/sec), and a limiting acceleration (volts/sec^2). If T is the applied torque, θ_o the output displacement angle, I the inertia of the mechanism, and f the viscous friction,

$$T = I\ddot{\theta}_o + f\dot{\theta}_o \tag{31}$$

But the specifications are in the form

$$(\dot{\theta}_o)_{max} = \frac{T_{max}}{f} \quad ; \quad (\ddot{\theta}_o)_{max} = \frac{T_{max}}{I}$$

so it is possible to be within the specified velocity and acceleration limits and still exceed the servo motor's capabilities because the combined torque requirements are greater than T_{max}.

If the input to the servo shaft is a sine wave of known magnitude and frequency, it is fairly easy to determine whether or not the input is beyond the capabilities of the servo. If the input is more complex, however, the only information available may be an estimate of the bandwidth of the input signal and the power per unit bandwidth of the input signal.

If the effect of the drifts and biases throughout the circuitry is to be kept to a minimum, the signal level should be kept as high as possible. A calculation of the maximum allowable signal can be made if an estimate of the filtering properties of the system between the input and the servo shaft is available. Thus if $H^*(s)$ is the estimated approximation of $H(s)$, and $W(\omega)$ is the noise power per unit bandwidth, the square of the standard deviation of f_o in terms of position, velocity, and acceleration may be found from

$$\sigma_p^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} W(\omega)\,|H^*(\omega)|^2\, d\omega \ , \tag{32}$$

$$\sigma_v^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} W(\omega)\,|H^*(\omega)|^2\,\omega^2\,d\omega\,, \quad \text{and}$$

$$\sigma_a^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} W(\omega)\,|H^*(\omega)|^2\,\omega^4\,d\omega$$, respectively.

Quite often the assumption that

$$W(\omega) = W \quad \text{for} \quad -\omega_c < \omega < \omega_c$$

$$= 0 \quad \text{elsewhere}$$

may be made without greatly harming the accuracy of the calculations, and thus the integrals above become

$$\sigma_p^2 = \frac{W}{2\pi}\int_{-\omega_c}^{\omega_c} |H^*(\omega)|^2\,d\omega$$

$$\sigma_v^2 = \frac{W}{2\pi}\int_{-\omega_c}^{\omega_c} |H^*(\omega)|^2\,\omega^2\,d\omega \tag{33}$$

$$\sigma_a^2 = \frac{W}{2\pi}\int_{-\omega_c}^{\omega_c} |H^*(\omega)|^2\,\omega^4\,d\omega$$

which may be evaluated by numerical methods to give these deviations as a function of ω_c and W, the noise power per unit bandwidth. If the variances obtained are multiplied by four and set equal to the respective servo limits, values of ω_c and W will be found which, for Gaussian distributions, will saturate the servo less than .006% of the time, thereby eliminating a potential source of inaccuracy.

Effect of Finite Averaging Times

As the voltage to be averaged in the mechanization of this method of system analysis is the product of two essentially random functions, the product is random itself, and a statistical description is the best that could be expected. It can be shown that the expected deviation is given by

$$\frac{\sigma_{a_n}^2}{a_n^2} = \frac{1.5\, T_1}{T} \tag{34}$$

where T_1 is the time constant of the approximating function, and T is the averaging time in seconds.

The maximum averaging time allowable is a function of the multiplier output level and the saturating voltage of the integrator used in the averaging. To minimize the effects of the biases previously examined, the signal level should be as high as the servo multiplier limitations will permit. Since it can be assumed that the errors due to finite averaging times are random, large equivalent averaging times may be attained by summing the voltages and times for many short runs.

COMPUTER MEASUREMENTS

Computer work connected with this study was undertaken for the purpose of:

1. Verifying the analytical development.
2. Investigating difficulties which may be encountered in the application of the theory.
3. Verifying the error analysis.
4. Reaching a conclusion as to the feasibility of the proposed method of measuring system transfer functions.

The computer laboratory work may be divided into four main efforts:

1. Setting up and calibrating the noise source.
2. Mechanizing the orthonormal approximating filters.
3. Mechanizing the systems to be studied and obtaining the required coefficients.
4. Simulating the systems under study with the approximating filters.

Computer equipment available included two drift stabilized 16 amplifier Michigan analog computers and one portable 20 amplifier Michigan analog computer.

Noise Source

Because it greatly simplifies computation, white noise was used throughout as the input for the analog computer simulation. White noise may be defined as a signal with constant power per unit bandwidth (W) over all frequencies. Physical limitations require that a real white noise source be limited in bandwidth. The assumption is made that as long as the power per unit bandwidth is constant over considerably more than the frequency range of interest in a particular problem, then the noise may be considered white for that problem.

In the computer work for this study, systems under test were used which had cut off frequencies near one cycle per second. With this in mind it was considered sufficient to supply a noise source with an upper cut off frequency of approximately 3 cycles per second.

The General Radio noise generator used procudes Gaussian white noise over a frequency range from 30 cycles per second to 20, 000 cycles per second. In order to obtain the desired input frequencies, the noise generator output was put through a gating and filter network, mechanized on a Michigan 20 amplifier

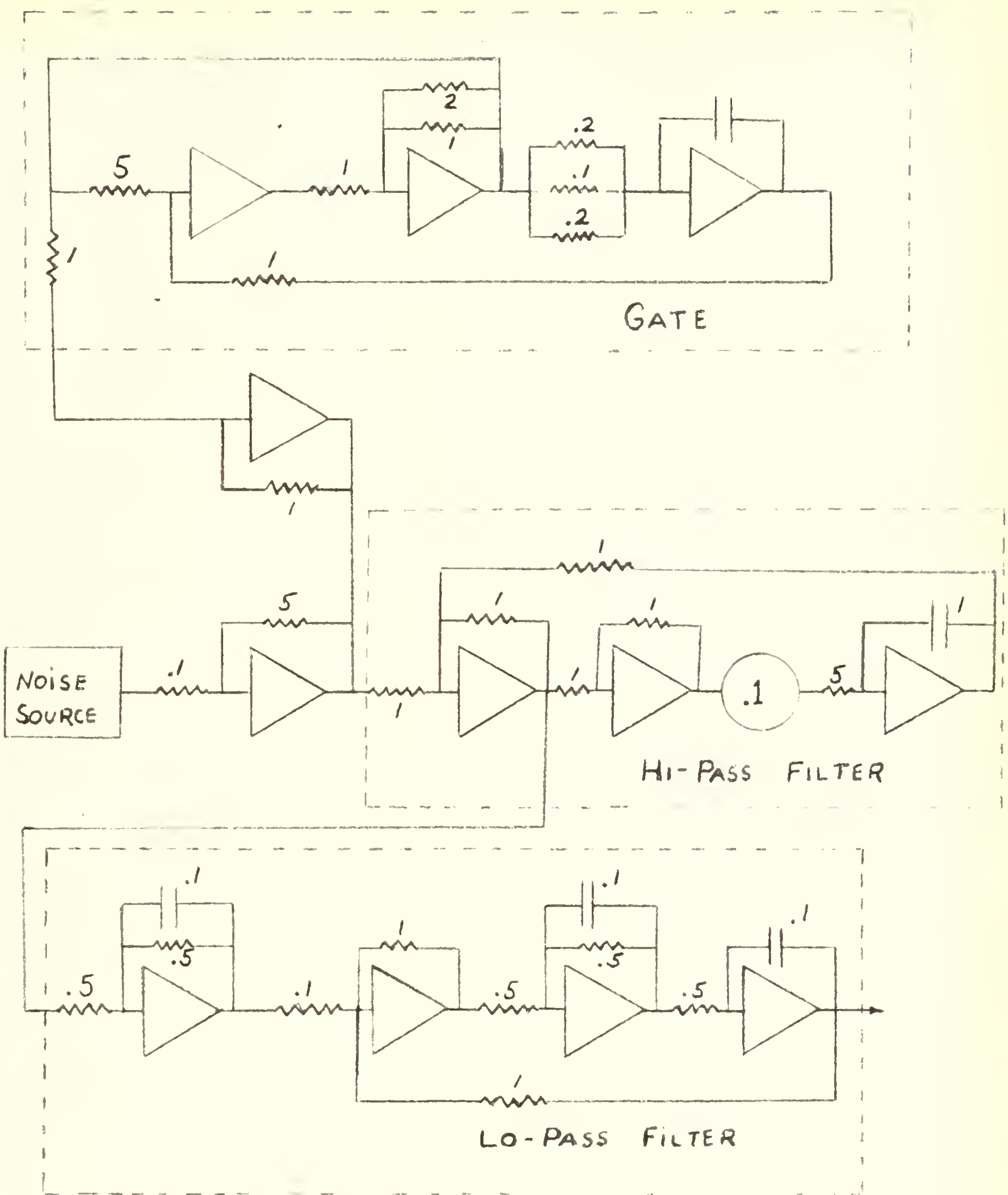

FIGURE 5

Noise Filtering Circuit.

computer. As shown in Fig. 5 the noise was first gated at approximately 200 cycles per second, which by sampling theory introduces beat frequencies in the lower end of the frequency spectrum. This gated signal was then sent through a high pass filter with a lower cut off frequency of .02 radians per second, which removed the DC bias caused by gating. Finally a third order Butterworth low pass filter with a cut off frequency of 20 radians per second attenuated the higher frequency power. The resulting noise source was essentially of constant power per unit bandwidth over the frequency range which was used.

Under the assumption that the power per unit bandwidth was constant, its value was measured as shown schematically in Fig. 6. The procedure was to obtain the total power output of the low pass filter shown, by squaring and averaging its output. This total power may be equated to the power per unit bandwidth (W) at a certain frequency, multiplied by the filter gain squared at that frequency, and integrated over all frequencies. Since the power per unit bandwidth is assumed constant, the total power integral may be written as

$$\text{Total Power} = \frac{W}{2\pi}\int_{-\infty}^{+\infty} \frac{1}{1+\tau^2\omega^2}\, d\omega \tag{35}$$

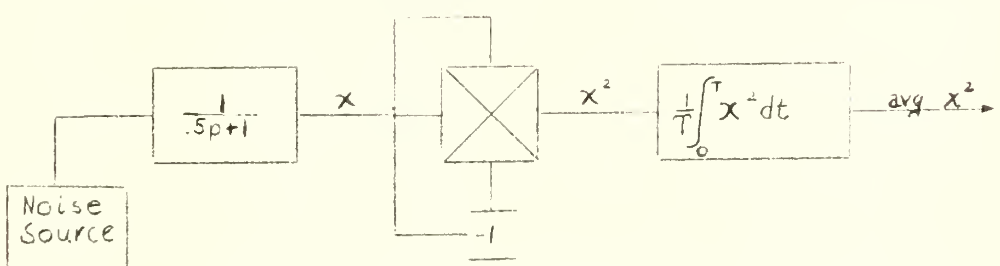

FIGURE 6

Noise Power Per Unit Bandwidth Measurement

Equating the total power as measured in Fig. 6 to the total power from Eq. 35;

$$avg\ x^2 = \frac{W}{2\pi}\int_{-\infty}^{+\infty}\frac{d\omega}{1+.25\omega^2} = \frac{W}{2\pi}\left[\frac{1}{.5}\left\{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right\}\right] = W \tag{36}$$

Because a significant time variation in noise source voltage was noticed, a power measurement was made with each averaging run throughout this study.

Approximating Functions

A set of four second order orthonormal approximating functions was mechanized as shown in Fig. 7. The four functions were:

$$\Phi_1(s) = \frac{2\sqrt{.4}\ s}{s^2+.8s+1} \qquad \Phi_3(s) = \frac{(s+.4)^2+1}{(s-.4)^2+1}\cdot\frac{2\sqrt{.8}\ s}{s^2+1.6s+1}$$

$$\Phi_2(s) = \frac{2\sqrt{.4}}{s^2+.8s+1} \qquad \Phi_4(s) = \frac{(s+.4)^2+1}{(s-.4)^2+1}\cdot\frac{2\sqrt{.8}}{s^2+1.6s+1} \tag{37}$$

in which

$$\alpha_1 = .4\ ,\ \beta_1 = 1 \quad ; \quad \alpha_3 = .8\ ,\ \beta_3 = 1$$

The weighting function of each of the above was obtained by supplying an impulse to the system. These weighting functions are plotted in Figs. 8 and 9. A schematic representation of how these functions were used is shown in Fig. 10. Note that in measuring a system function in accordance with the theoretical development, it is intended to have the system output on the shaft of the servo multipliers. However, it was desirable here to first verify the orthonormality of the approximating functions, and in so doing obtain a feeling for the expected

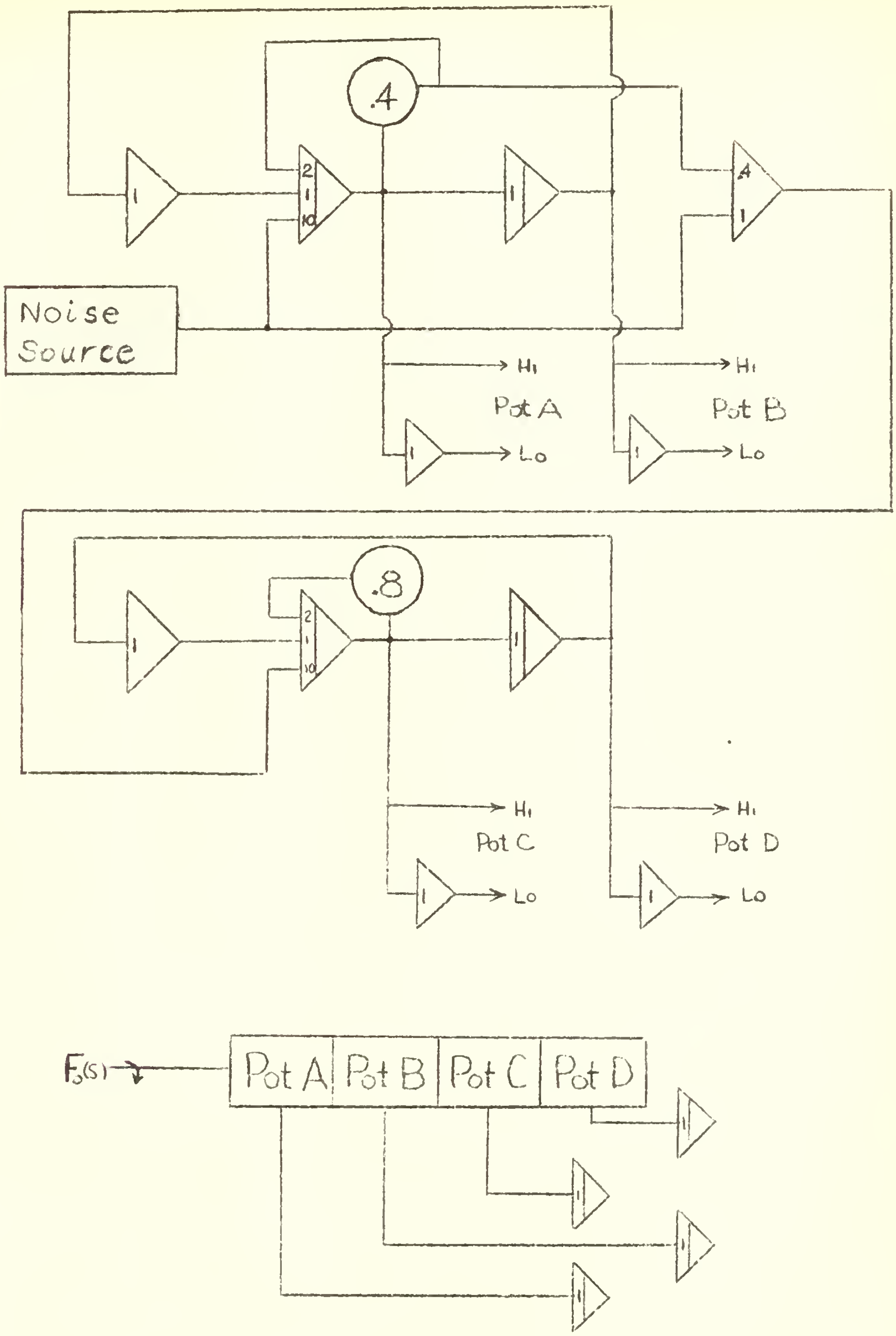

FIGURE 7

Orthonormal Approximating Function Circuit

accuracy of the procedure. This simply involved using the output of one of the orthonormal filters to drive the servo shaft, i.e. $h(t) = \varphi_n(t)$, and putting all four filters on the servo potentiometers. Eqs. 17 and 18 show that the coefficient of the function fed to the shaft, multiplied by itself, integrated, and averaged, should be the noise per unit bandwidth, W. Eqs. 17 and 18 show that the other three coefficients should be zero. Averaging times of 1000 seconds were used in these runs, and the results, normalized with respect to W, are tabulated in Table I. For the case where the output of Φ_1 (s) is on the shaft, Fig. 11 is a plot of error in experimental results against time, showing error as a percentage of a_1.

Table I.
Experimental Coefficients for Orthonormal Check of
Approximating Functions

	Functions Replacing H(s)			
	$\Phi_1(s)$	$\Phi_2(s)$	$\Phi_3(s)$	$\Phi_4(s)$
a_1	.985	.005	.029	.007
a_2	.003	.962	.023	.070
a_3	.019	.148	.922	.008
a_4	.028	.009	.008	.984

Static error measurements were read as

$$k_1 = .16 \text{ Volts}, \quad k_2 = .15 \text{ Volts}, \quad k_3 = .15 \text{ Volts} \quad \& \quad k_4 = 2 \times 10^{-4} \text{ Volts}.$$

Using these static errors in Eq. 29, and an averaging time of 1000 seconds in Eq. 34, a predicted error of 5% was obtained. The results of Table I seem to conform fairly well to this predicted error.

Systems Under Study

Since this work is limited to a feasibility study, no unknown system was investigated. Rather two known systems were used for H(s), so that the quality of the approximation could be checked. The systems were

$$H_1(s) = \frac{.5}{s^2+s+.5} \quad ; \quad H_2(s) = \frac{s-.5}{s^2+s+.5} \tag{38}$$

Note that the poles of these systems are, in s-plane representation, at $-.5 \pm j.5$, as compared to $-.4 \pm j.92$ and $-.8 \pm j.6$ for the approximating functions. Because the approximating functions have no poles coincident with the systems under study, and a finite number of approximating functions are being used, there must be some error in the approximations. The object was to find the set of coefficients which gave the least mean square error approximation to each of the systems.

The desired coefficients were found analytically by the method of residues. Evaluation of the residues resulted in columns one and three of Table II. Columns two and four of this table contain the coefficients found experimentally using the circuit shown schematically in Fig. 10.

Table II.
Calculated and Experimental Coefficients for $H_1^*(s)$ and $H_2^*(s)$

	$H_1(s)$		$H_2(s)$	
	Calculated	Experimental	Calculated	Experimental
a_1	-.145	-.137	.971	1.0
a_2	.521	.568	-.228	-.209
a_3	-.092	-.016	-.182	-.158
a_4	-.456	-.476	-.660	-.584

After the coefficients were determined, the original systems were simulated by the approximating filters, weighted properly. Figs. 12 and 13 show the weighting functions of the linear second order systems, the simulated weighting functions using analytically determined coefficients, and the simulated weighting functions using experimentally determined coefficients. Note that the most serious error in both cases was caused by the poor correspondence between the approximating filters and the systems being studied. A greater number of approximating functions, or adjustment of the poles of these functions, would reduce this error. The error due to finite averaging times and component inaccuracies was comparatively small; that is, the calculated and experimental approximations are fairly close to each other.

Limitations

The computer work brought out some serious limitations on this method of measuring system transfer functions.

1. Electronic error considerations showed the necessity of using only drift stabilized amplifiers both in the approximating filter circuit and as averaging integrators.
2. The most severe limitation was imposed by the servo multiplier. The cut off frequency of the system under test had to be kept at approximately five cycles per second or less in order not to exceed servo velocity limitation. The velocity limitation also dictated that the filtered noise source be limited to a fairly narrow bandwidth. A twenty radian per second bandwidth was used successfully in this work. A two hundred radian per second bandwidth was tried but was found to cause servo

overload by exceeding the velocity power limit. Because the systems were limited to low cut off frequencies, long averaging times were required.

3. The question of pole optimization in choosing approximating functions was not considered in this study. It should be noted that with the arbitrary choice of poles made here, the approximation is of little real use. For accurate approximation a better choice of pole locations, coupled with an increase in the number of the approximating functions, is necessary.

CONCLUSIONS

The proposed method of measuring system transfer functions was found to be feasible within limits. In order to make accurate measurements on an actual unknown system, the electronic equipment available must include drift stabilized amplifiers and electronic multipliers capable of handling multiple inputs. The number of amplifiers required is dependent upon the knowledge available on the system under study and its complexity. Some knowledge of the system is definitely desirable.

Once the coefficients have been determined, the system under study is easily simulated by the approximating network. In this manner an electronic approximation to the system under study is immediately available as a component in an electronic circuit.

Although the theory holds equally well for an arbitrary input and for general approximating functions, this study has been limited to white noise input and

orthonormal approximating functions in order to illustrate the theory involved and to carry this theory through a practical development. It is believed that the results justify further efforts along this line.

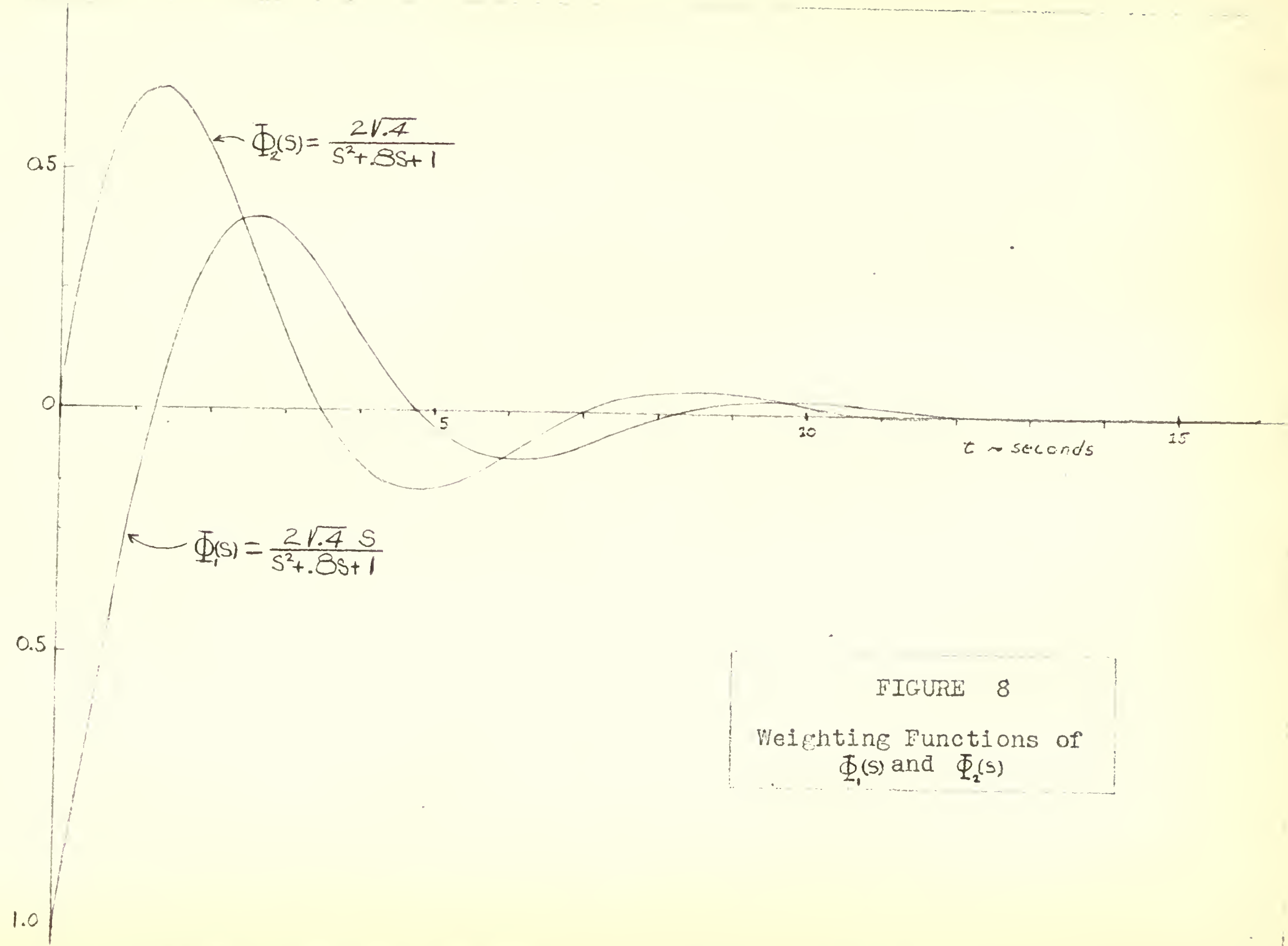

FIGURE 8

Weighting Functions of $\Phi_1(s)$ and $\Phi_2(s)$

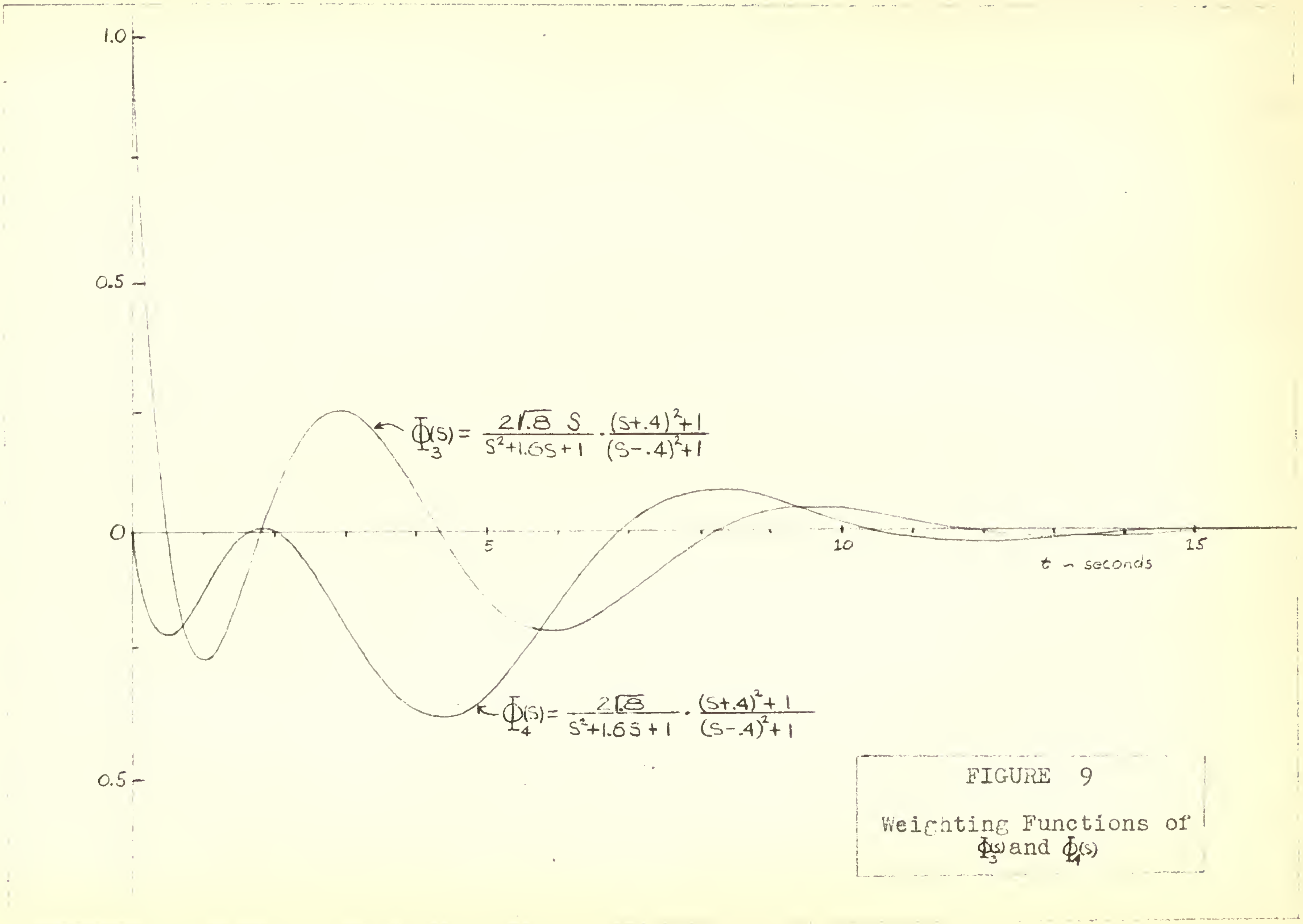

FIGURE 9

Weighting Functions of $\Phi_3(s)$ and $\Phi_4(s)$

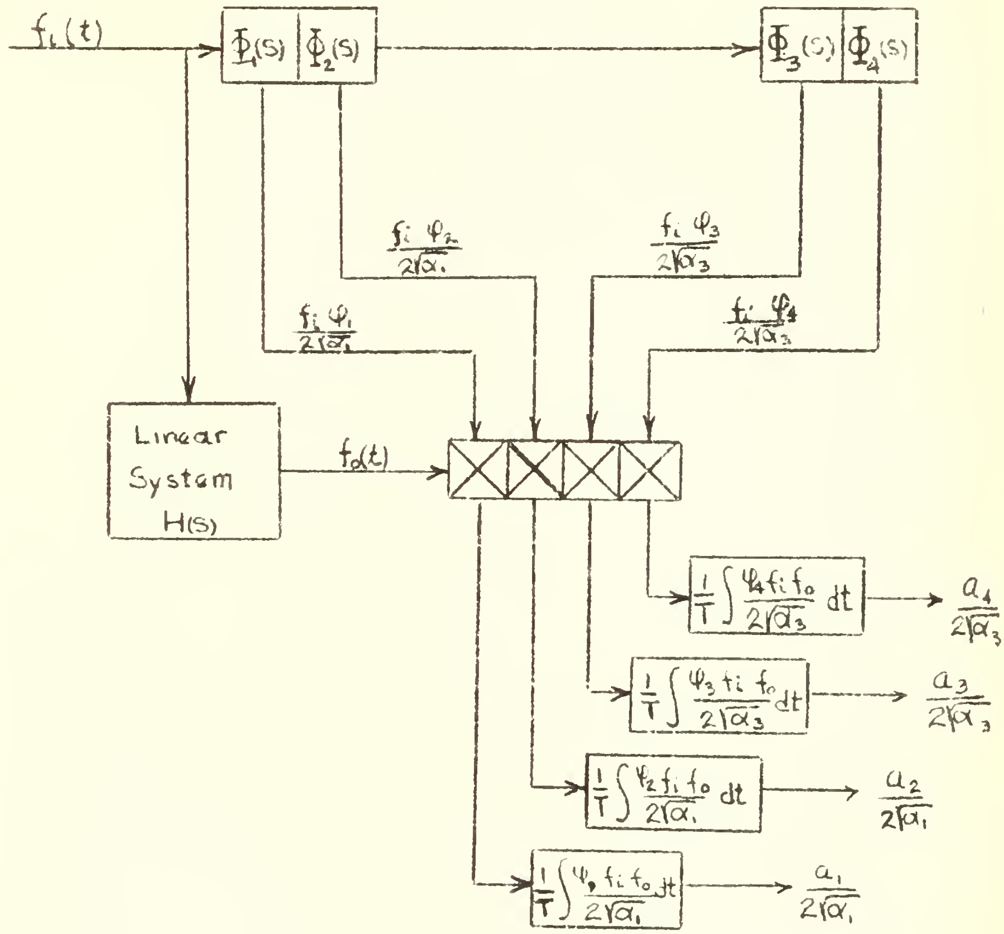

FIGURE 10

Procedure Used in Obtaining Coefficients of Approximating Functions

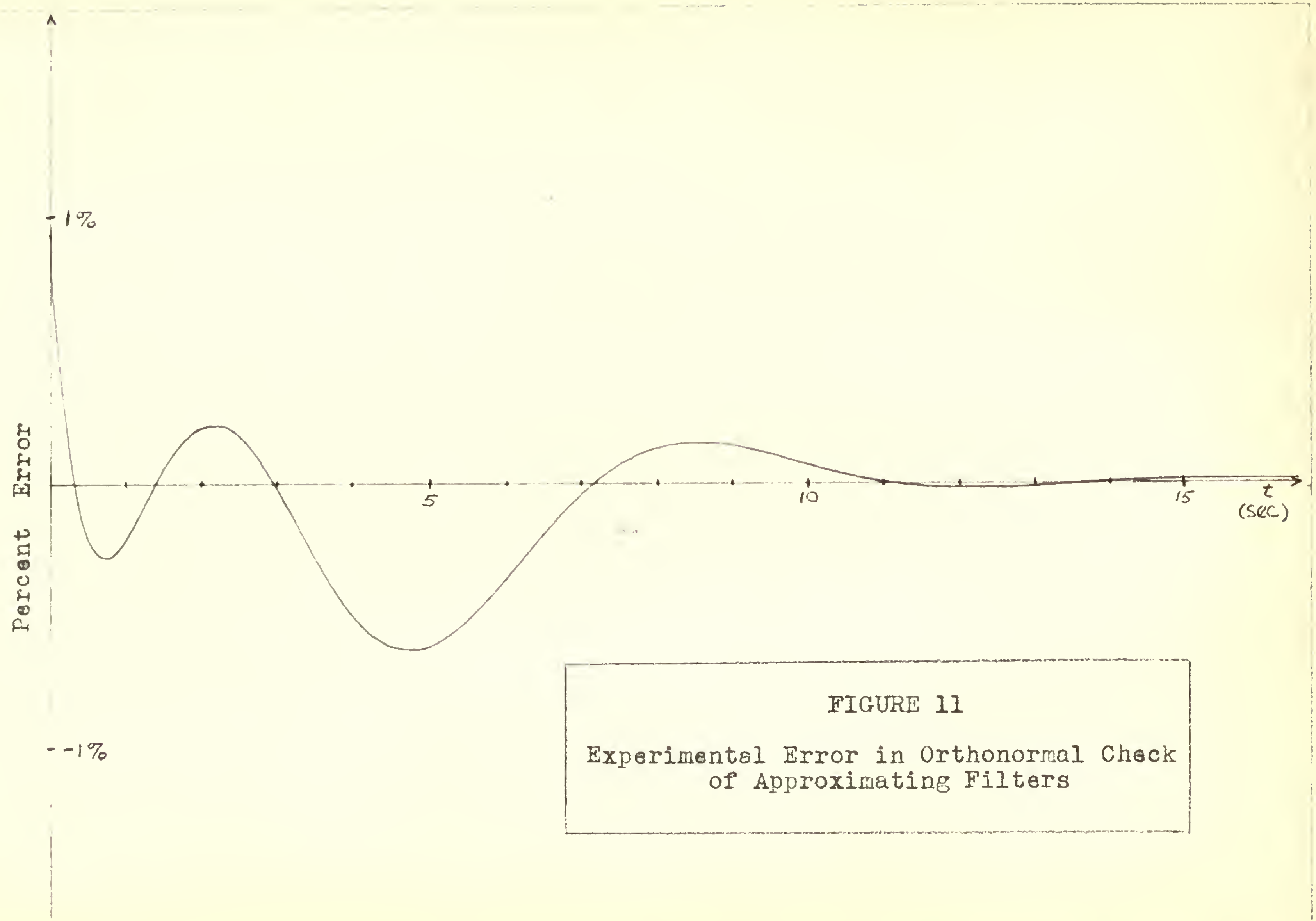

FIGURE 11

Experimental Error in Orthonormal Check of Approximating Filters

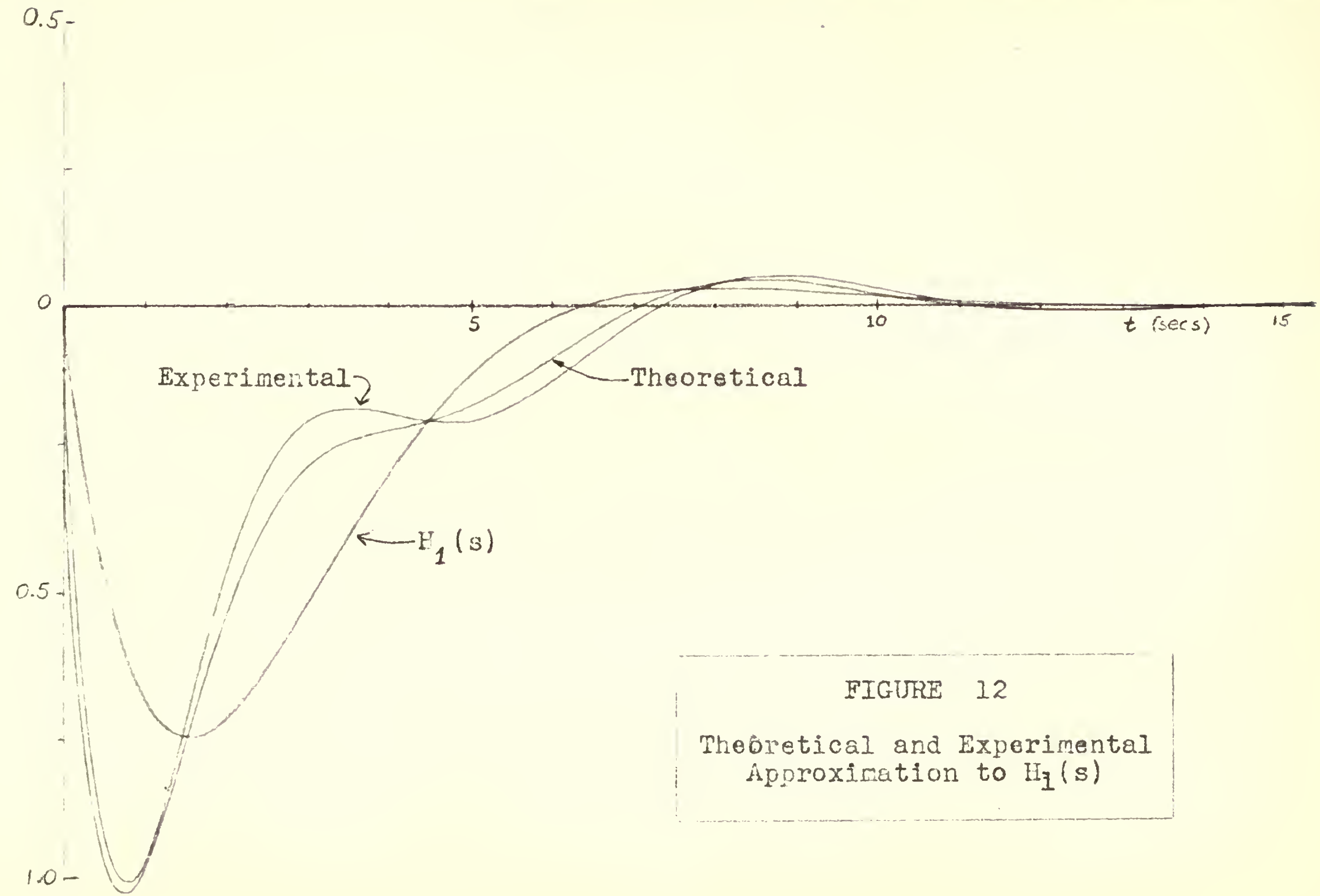

FIGURE 12

Theoretical and Experimental Approximation to $H_1(s)$

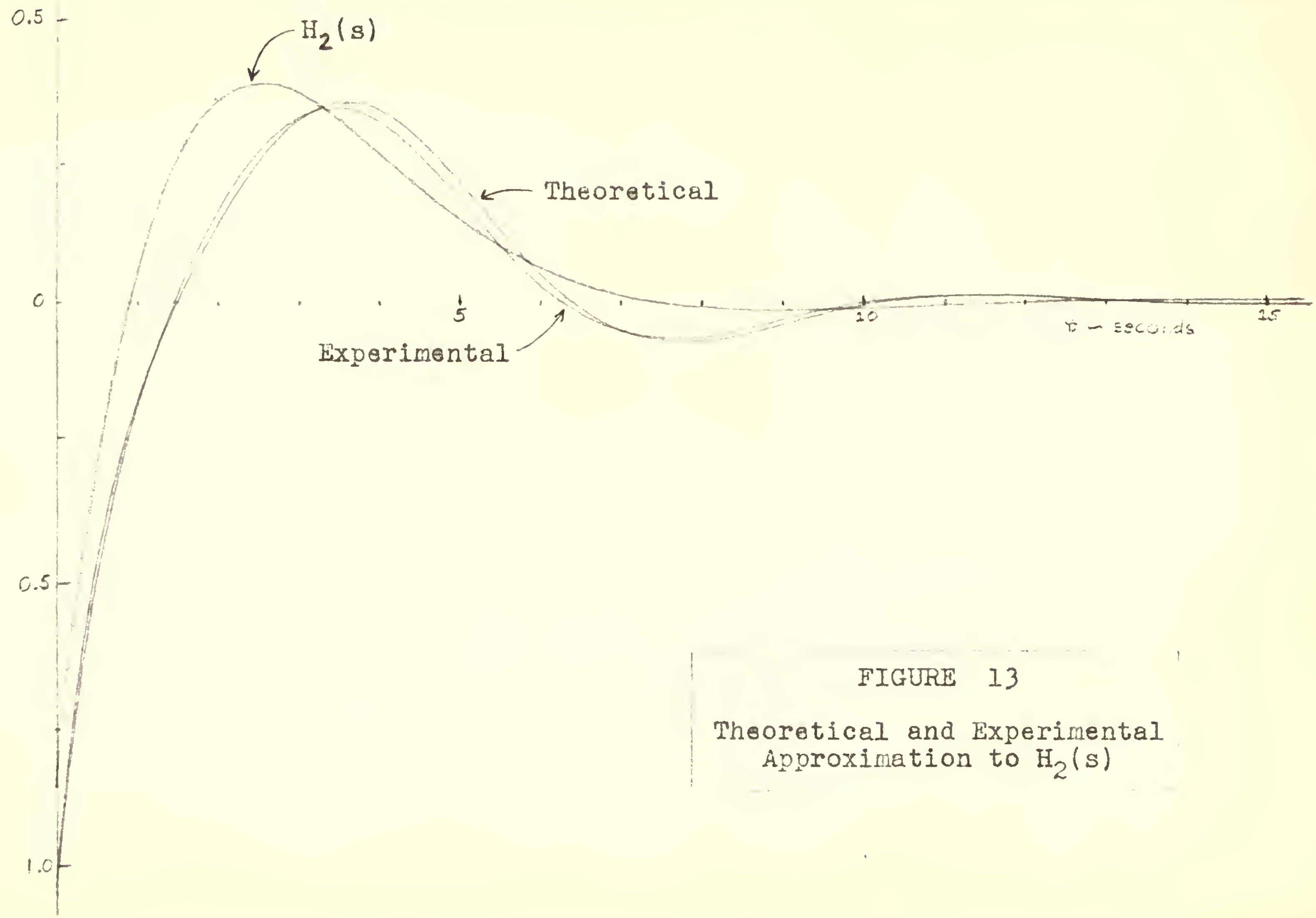

FIGURE 13

Theoretical and Experimental Approximation to $H_2(s)$

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